# E.N.A.C 2003

$E = \mathcal{A}(\mathbb{R})$ ; $E_1 = \mathcal{C}(\mathbb{R})$

$\varphi \in E_1$ ; $g$ définie par $\forall x \in \mathbb{R}\quad g(x) = \int_0^x \varphi(t) f(t)\,dt \quad (f \in E_1)$

$L_\varphi : f \longmapsto g$

$f$ étant continue sur $\mathbb{R}$, $\varphi . f$ aussi et $g$ est la primitive de $\varphi . f$ sur $\mathbb{R}$ qui s'annule en $0$

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**Question 1** : $\varphi$ est donnée dans $E_1$.

$g \in \operatorname{Im} L_\varphi \iff \exists f \in E_1 \quad \forall x \in \mathbb{R} \quad g(x) = \int_0^x \varphi(t) f(t)\,dt$

$\Rightarrow \exists f \in E_1 \quad g' = \varphi . f$

. donc * si $\varphi$ ne s'annule pas, il existe $f$, $f \in E_1$ telle que $\dfrac{g'}{\varphi} = f$ $\boxed{a}$ $\boxed{b}$

* si $\varphi$ s'annule en $\alpha$ par exemple,

on a $g'(\alpha) = 0$ et pour $x$ tel que $\varphi(x) \neq 0$, $\dfrac{g'(x)}{\varphi(x)} = f(x)$ donc $\dfrac{g'}{\varphi}$ est prolongeable par continuité en ces points car $f \in E_1$

on ne sait pas si $\varphi$ s'annule en des valeurs "isolées" ou non

* si $\varphi = \tilde{0}$ , $g'/\varphi$ n'existe pas

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**Question 2** $\rightarrow$ $\varphi(x) = \sqrt{1+x^2} - 1 \qquad g(x) = e^x - x - 1$

$g$ est bien continue, ainsi que $g'$ $(\forall x \quad g'(x) = e^x - 1)$

Pour $x \neq 0$ $\quad \dfrac{g'(x)}{\varphi(x)} = \dfrac{e^x - 1}{\sqrt{1+x^2} - 1} = \dfrac{x + o(x)}{\frac{1}{2}x^2 + o(x^2)} = \dfrac{1 + \varepsilon(x)}{\frac{1}{2}x + o(x)}$ pas de limite finie en $0$ $\boxed{a}$

$\rightarrow$ $\varphi(x) = \ln(1 + x + x^2) \qquad g(x) = \dfrac{x^2}{2} + \dfrac{x^3}{3} \qquad g'(x) = x + x^2$

$1 + x + x^2 > 0 \quad \forall x$ donc $\varphi \in E_1$

$1 + x + x^2 = 1 \iff x = 0$ ou $x = -1$

- $x \neq 0 \quad \dfrac{g'(x)}{\varphi(x)} = \dfrac{x + x^2}{\ln(1+x+x^2)} = \dfrac{x + x^2}{x + x^2 - \frac{1}{2}x^2 + o(x^2)} = \dfrac{1+x}{1 + \frac{1}{2}x + o(x)} \rightarrow 1$ si $x \rightarrow 0$

- $x \neq -1 \quad \dfrac{g'(x)}{\varphi(x)} = \dfrac{(t-1)t}{\ln(1-t+t^2)} = \dfrac{-t + t^2}{-t + t^2 - \frac{1}{2}t^2 + o(t^2)} = \dfrac{-1+t}{-1 + \frac{1}{2}t + o(t)} \rightarrow +1$ si $t \rightarrow 0$

$t = x + 1$ $\boxed{c}$

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**Question 3** $\quad \forall x \quad g'(x) = \varphi(x) f(x)$ . $\boxed{\varnothing}$

On ne sait rien sur $\int_0^x \varphi(t)\,dt$ donc on ne peut diviser... $\varphi$ peut s'annuler

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**Question 4** $\quad \varphi(x) = \ln(1 + x + x^2)$ et $g(x) = \dfrac{x^2}{2} + \dfrac{x^3}{3} \qquad g' = \varphi . f$

On a vu à la question 2 que on peut poser $f(0) = 1$ et $f(-1) = 1$ $\boxed{b}$

## Question 5

si $x \notin \{0,-1\}$ $\quad f(x) = \dfrac{x+x^2}{\ln(1+x+x^2)} \qquad f(0)=1\ ;\ f(-1)=1$

$$f(x)+f(-1-x) = \frac{x+x^2}{\ln(1+x+x^2)} + \frac{(-1-x)+(-1-x)^2}{\ln(1+x^2+x)} = 2f(x) \quad \text{si } x\neq 0,-1$$

$-1 \qquad -1/2 \qquad 0$

$\forall x \quad f(-1-x)=f(x)$ $\qquad$ symétrie % $(x=-1/2)$ $\qquad \boxed{\emptyset}$

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## Question 6

$\boxed{c}$ $\quad \boxed{d}$

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## Question 7

$x>0 \quad \dfrac{f(x)}{x} = \dfrac{1+x}{\ln(1+x+x^2)} \to +\infty$ $\qquad \boxed{d}$

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## Question 8

$A(0,1)$

$$\frac{f(x)-f(0)}{x} = \frac{f(x)-1}{x} = \frac{1+x}{\ln(1+x+x^2)} - \frac{1}{x} = \frac{1+x}{x\left(1+\frac12 x+o(x)\right)} - \frac1x$$

$$= \frac1x\left[(1+x)\left(1-\frac12 x+o(x)\right)-1\right] = \frac1x\left[\frac12 x+o(x)\right] = \frac12+\varepsilon(x) \xrightarrow[x\to 0]{} 1/2$$

Tgte en A : $y-1=\frac12 x$ $\qquad \boxed{c}$

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*Cas général* $\qquad \Phi(x)=\int_0^x \varphi(t)\,dt$

## Question 9

$g=L_\varphi(f)$ et $g-f=\Phi$ ??

* on a tjs $g-f=\Phi$

a) $g'(x)=\varphi(x)+2\varphi(x)e^{\Phi(x)} = \varphi(x)f(x)$

$\quad g(0)=\Phi(0)+1+2e^{\Phi(0)}=4$

b) $g'(x)=-\varphi(x)e^{\Phi(x)}+\varphi(x)=\varphi(x)f(x)$ $\quad$ donc $g=L_\varphi(f)$ $\qquad \boxed{b}$

$\quad g(0)=0$

c) $\Phi(0)=0$ donc $g$ non définie en 0

d) $g(0)=3$

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## Question 10

$\varphi(x)=e^x \qquad E_3=\{f \mid \exists(a,b)\in\mathbb{R}^2\ \forall x\ f(x)=a\cos x+b\sin x\}$

$$\int_0^x \varphi(t)(a\cos t+b\sin t)\,dt = \frac{a-b}{2}e^x\cos x+\frac{a+b}{2}e^x\sin x-\frac a2+\frac b2$$

$$= \frac a2\left(e^x\cos x+e^x\sin x-1\right)+\frac b2\left(-e^x\cos x+e^x\sin x+1\right)$$

$\boxed{b}$

## Question 11

$f_1(x) = \cos x \; ; \; f_2(x) = \sin x$

$$L\varphi(f_1)(x) = e^x \frac{\cos x + \sin x}{2} - \frac{1}{2} = V_2(x) = \frac{u_1(x) + u_2(x)}{2}$$

$$L\varphi(f_2)(x) = e^x \frac{\sin x - \cos x}{2} + \frac{1}{2} = V_1(x) = \frac{u_2(x) - u_1(x)}{2}$$

$\boxed{c}$

## Question 12

$u_n(x) = x^n e^x \quad (n \geqslant 1) \; ; \; u_0(x) = e^x - 1$

$$\int_0^x t^p e^t \, dt = g_p(x) \qquad p \geqslant 1$$

• $\begin{cases} u'(t) = e^t \\ v(t) = t^p \end{cases}$ $\quad \begin{cases} u(t) = e^t \\ v'(t) = p t^{p-1} \end{cases}$ $\quad g_p(x) = x^p e^x - p \, g_{p-1}(x)$

pour $p = 0$ $\quad g_0(x) = \int_0^x e^t \, dt = e^x - 1 = u_0(x)$

Donc $\boxed{b}$ et $\boxed{d}$

## Question 13

$g_0 = u_0 \qquad \forall p \geqslant 1 \quad g_p = u_p - p \, g_{p-1}$ $\qquad \frac{p!}{k!} = A_p^{p-k}$

$g_0 = u_0$

$g_1 = u_1 - g_0$

$g_2 = u_2 - 2 g_1$

$g_3 = u_3 - 3 g_2$

$g_{p-1} = u_{p-1} - (p-1) g_{p-2}$

$g_p = u_p - p \, g_{p-1}$

$$g_p = \sum_{k=0}^{p} (-1)^{p-k} A_p^{p-k} u_k$$

$\boxed{c}$

$$g_{p+1} = u_{p+1} - (p+1) \sum_{k=0}^{p} (-1)^{p-k} A_p^{p-k} u_k = \sum_{k=0}^{p+1} (-1)^{p+1-k} A_{p+1}^{p+1-k} u_k$$

$$\left( A_{p+1}^{p+1-k} = (p+1) A_p^{p-k} \right)$$

## Question 14

b.c de $P_n$ : $(1, x, x^2, \ldots, x^n)$ $\qquad \mathcal{B} = (u_0, \ldots, u_n)$

$$\text{Mat } L\varphi_n = \begin{pmatrix} 1 & -1 & \times & \times & \times \\ 0 & 1 & & \times & \times \\ \vdots & 0 & 1 & & \times \\ \vdots & \vdots & & \times & \times \\ \vdots & \vdots & & & \times \\ 0 & 0 & & & 1 \end{pmatrix}$$

$\boxed{\emptyset}$

## Partie II

$$P = 26 - 34X + 23X^2 - 6X^3 + X^4$$

$$P = (X-2-3i)(X-2+3i)(X-1-i)(X-1+i)$$
$$= (X^2-4X+13)(X^2-2X+2)$$

$$\frac{1}{P} = \frac{2X+7}{85(X^2-2X+2)} + \frac{-2X-3}{85(X^2-4X+13)}$$

### Question 15

$x = 1$ est la seule racine commune $\boxed{d}$

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### Question 16

$\boxed{b}$

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### Question 17

$\boxed{c}$

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### Question 18

$\boxed{b}$

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### Question 19

$$\int_0^A \frac{dx}{P(x)} = \frac{1}{85} \ln \frac{A^2-2A+2}{A^2-4A+13} - \frac{7}{255} \operatorname{Arctan} \frac{A-2}{3} + \frac{9}{85} \operatorname{Arctan}(A-1)$$
$$+ \frac{12\ln 13 - 12 \ln 12 - 28 \operatorname{Arctan}(2/3) + 27\pi}{1020}$$

$\frac{27}{1020} = \frac{9}{340} = \frac{9}{85 \times 4}$ $\quad 85 \times 12 = 1020$ $\boxed{d}$

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## Partie III

$f(0) = 0$, $f(1) = -1$, $f(-1) = 1$ $\quad f(t) = \frac{2t}{1-t^2} \ln|t|$ sinon

### Question 20

a) est obligat$^t$ faux psq $f(1) \neq f(-1)$

c) est stupide $\boxed{b}$

d) aussi ...

b) ?? $\forall t \notin \{0,-1,1\}$ $\quad f(-t) = -f(t)$ et OK pr $0, 1, -1$

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### Question 21

a) stupide $\boxed{c}$

Qd $t \to 0$ $\quad t \ln|t| \to 0$ : continuité en $0$ psq $f(0) = 0$

Qd $t \to 1$ $\quad \frac{\ln t}{t-1} \to 1$ $\quad -\frac{2t \ln t}{(t-1)(t+1)} \to 1 \times -1 = f(-1)$ continuité en $1$ donc en $-1$

$\boxed{d}$ aussi ...

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### Question 22

$\ln|t| \underset{+\infty}{=} o(t)$ donc $\lim_{\infty} f = 0$ $\boxed{c}$

## Question 23

$t \neq 0$ , $t \neq 1$ et $-1$ $\quad f\left(\frac{1}{t}\right) = \dfrac{\frac{2}{t}}{1-\frac{1}{t^2}} \ln \frac{1}{|t|} = \dfrac{2t}{t^2-1}\left(-\ln|t|\right) = f(t)$ $\quad \boxed{b}$

$f\left(\frac{1}{1}\right) = f(1)$ ; $f\left(\frac{1}{-1}\right) = f(-1)$ $\quad \boxed{d}$

## Question 24

$\frac{1}{t} f(t) = \dfrac{2}{1-t^2} \ln|t| \xrightarrow[t\to 0]{} -\infty$ $\quad \boxed{c}$

## Question 25

$\underset{t = x+1}{t \ln t} = (x+1)\ln(1+x) = (x+1)\left(x - \frac{x^2}{2} + \dots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)\right)$

$= x - \frac{x^2}{2} + x^2 + \frac{x^3}{3} - \frac{x^3}{2} + \dots (-1)^{k-1}\frac{x^{k+1}}{k} + (-1)^k \frac{x^{k+1}}{k+1} + \dots$

$+ (-1)^{n-1}\frac{x^n}{n} + (-1)^{n-2}\frac{x^n}{n-1} + o(x^n)$

$t\ln t = (t-1) + \frac{1}{2}(t-1)^2 - \frac{1}{6}(t-1)^3 + \dots + (-1)^n\left(\frac{1}{n-1} - \frac{1}{n}\right)(t-1)^n + o(t-1)^n$ $\quad \boxed{d}$

$\frac{1}{-1} - \frac{1}{n} = \frac{1}{n(n+1)}$

## Question 26

$t \neq 1$ au $V(1)$ $\quad \dfrac{f(t)+1}{t-1} = \dfrac{-2t}{(t-1)^2(t+1)}\ln t + \dfrac{1}{t-1} = \dfrac{2t\ln t + 1 - t^2}{(1-t^2)(t-1)}$ $\quad 1-t^2 = (1-t)(1+t) = -x(x+2)$

$t = x+1$

$= \dfrac{2(x+1)\ln(x+1) - x(x+2)}{-x^2(x+2)} = \dfrac{2(x+1)\left(x - \frac{x^2}{2} + o(x^2)\right) - x(x+2)}{-x^2(x+2)}$

$= \dfrac{2\left(x + \frac{1}{2}x^2 + o(x^2)\right) - x^2 - 2x}{-x^2(x+2)} = \dfrac{o(x^2)}{-x^2(x+2)} = \dfrac{\varepsilon(x)}{x+2}$ donc $f'(1) = 0$

$= \dfrac{t-1}{6} - \dfrac{(t-1)^2}{6} + (t-1)^2\varepsilon(t)$ $\quad \boxed{c}$

## Question 27

$V(0)$ $\quad \dfrac{f(t)}{t} = \dfrac{2}{1-t^2}\ln|t| \to -\infty$ $\quad f$ non dér$^{ble}$ en $0$ $\quad \boxed{a}$ $\quad \boxed{d}$

$f'(-1) = 0 = f'(1)$ car $f'$ paire

## Question 28

$\varphi(t) = \ln t + \dfrac{1-t^2}{1+t^2}$

$\varphi'(t) = \dfrac{1}{t} - \dfrac{4t}{(1+t^2)^2} = \dfrac{(1+t^2)^2 - 4t^2}{t(1+t^2)^2} = \dfrac{(1-t^2)^2}{t(1+t^2)^2}$ $\quad \boxed{b}$ $\quad \boxed{d}$

## Question 29

$t \neq 0, 1, -1$ $\quad f'(t) = \dfrac{2(t^2+1)\ln|t|}{(t^2-1)^2} - \dfrac{2}{t^2-1} = \dfrac{2(1+t^2)}{(1-t^2)^2}\varphi(|t|)$

$\varphi \nearrow$ sur $\mathbb{R}_+^*$ et $\varphi(1) = 0$ donc $\boxed{b}$

## Question 30

$g$ impaire $\quad \boxed{d}$